Probing the Superfluid - Mott Insulating Shell Structure of Cold Atoms by Parametric Excitations

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We study the effect of parametric excitations on systems of confined ultracold Bose atoms in periodically modulated optical lattices. In the regime where Mott insulating and Superfluid domains coexist, we show that the dependence of the energy absorbed by the system on the frequency of the modulation serves as an experimental probe for the existence and properties of Mott insulating-Superfluid domains.

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The tunability of ultracold atom gases in optical lattices makes them ideal candidates for the investigation of quantum many-particle phenomena in different regimes. As a paradigmatic example, within the Bose-Hubbard (BH) model, bosonic atoms in optical lattices have been predicted to undergo a quantum phase transition from a compressible, phase-coherent superfluid (SF) phase to an incompressible incoherent Mott insulator (MI) induced by increasing the ratio of the on-site repulsion U to the hopping kinetic energy J [1]. For an infinite uniform system on a lattice the phase diagram in the $(J/U,\mu/U)$ plane is composed of lobe-like MI regions with integer site occupation in an otherwise SF phase [1, 2]. By controlling the ratio J/U via the intensity of the optical lattice, the MI-SF transition has been experimentally observed [3, 4] together with an energy gap of the expected magnitude in the MI phase. Experiments are always performed in finite inhomogeneous systems with a fixed total number of atoms subject to an external confinement which has been numerically established to induce the coexistence of MI and SF domains [5, 6, 7, 8].

The formation of MI shells of different site occupation have been recently detected [9] but still, an experimental probe of the co-existence of SF and MI domains and of their properties is presently missing. It has been proposed [10] that the presence of MI-SF domains modifies the low-energy excitation spectrum of the system, finally affecting its thermodynamic properties.

In this paper we analyze the energy absorption due to a periodic time modulation of the lattice intensity exploiting the parametric resonances of the modes of the system. We find this response to be a probe of the MI-SF domains, particularly at frequencies that correspond to energies lower than the MI gap. For these frequencies the energy absorption is only due to excitations of the SF regions, typically ring-like domains, and is strongly peaked at frequencies related to the transverse size of the SF rings. An experimental observation of these peaks may then provide evidence for the co-existence of MI-SF domains and give an estimate for their width.

A parametric excitation is a process in which the population of certain modes is exponentially enhanced as a result of a periodic temporal modulation of a parameter in the Hamiltonian. Parametric excitations have been studied theoretically in harmonically trapped BEC [11] and for condensates in optical lattices in different spatial dimensions [12]. Very recently a periodic variation of the optical lattice potential depth was used to probe the response of a 1D condensate across the MI-SF quantum phase transition [13, 14]. Here, the energy absorption was measured versus the optical lattice modulation frequency Ω, showing a broad spectrum in the SF phase and peaked features in the gapped MI one. Theoretical works that followed analyzed the parametric instabilities of Bogoliubov modes [15, 16]. In parallel, very recent works considered the linear response energy absorption of 1D systems within the bosonization picture [17] or with numerical diagonalization of realistic systems [18, 19, 20]. In the presence of confinement, the low-energy form of the dynamic structure factor has been argued to yield signatures of the presence of SF-MI domains [20].

The BH Hamiltonian describing a system of bosons in a periodic optical lattice is

$$\begin{aligned} H &= -J_0 \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + h.c. \right) + \frac{U_0}{2} \sum_i b_i^\dagger b_i \left(b_i^\dagger b_i - 1 \right) \\ &+ \sum_i (-\mu + V_i) b_i^\dagger b_i \end{aligned} \tag{1}$$

with b_i the annihilation operator of a boson at site i. We assume nearest-neighbour hopping of strength J_0, and an on-site repulsion U_0. The values of J_0 and U_0 are given by $J_0 = 4/\sqrt{\pi}(V_0/E_R)^{3/4}\exp[-4\sqrt{V_0/E_R}]E_R$ and $U_0 = \pi^2 a_0/3l(V_0/E_R)^{3/4}E_R$ [21], where V_0 and l are the optical lattice depth and spacing, E_R is the recoil energy and a_0 the scattering length. The chemical potential μ fixes the particle number and V_i is the potential bias at site i due to the external confinement.

We consider a two dimensional (2D) atom cloud subjected to an optical lattice. In the 2D plane the confining

potential V_i grows from the center of the cloud towards its boundaries. For J_0/U_0 larger than the critical value inducing the MI-SF transition, the entire 2D system is a SF while below this critical ratio concentric alternated 2D ring-like MI and SF domains are formed [5].

In a compressible SF ring the average occupation per site $\bar{n}$ is in between the integer occupations of the neighboring MI domains. Assuming the length of the ring is much larger than its width L, we can view it as a SF stripe closed with periodic boundaries and consider momentum eigenstates along the stripe. For weak depletion, in momentum space the BH Hamiltonian (1) in the SF stripe reduces, at mean field [22], to

$$H=\sum_{\mathbf{k}\neq 0}(\epsilon^0_{\mathbf{k}}+\bar{n}U_0)a^\dagger_{\mathbf{k}}a_{\mathbf{k}}+\frac{\bar{n}U_0}{2}\sum_{\mathbf{k}\neq 0}\left(a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}+a_{\mathbf{k}}a_{-\mathbf{k}}\right) \tag{2}$$

where the ground state energy was subtracted and $a_{\mathbf{k}}=\sum_j b_j e^{i\mathbf{k}\mathbf{r}_j}$. Within the tight-binding approximation (valid for the deep lattices implied in the MI domain formation) we have, at small k, $\epsilon^0_{\mathbf{k}}=J_0 l^2|\mathbf{k}|^2$.
The Hamiltonian (2) is diagonalized via the Bogoliubov transformation $c_{\mathbf{k}}=u_{\mathbf{k}}a_{\mathbf{k}}+v_{\mathbf{k}}a^\dagger_{-\mathbf{k}}$ with $u^2_{\mathbf{k}}=1+v^2_{\mathbf{k}}=\frac{1}{2}\left[\left(\epsilon^0_{\mathbf{k}}+\bar{n}U_0\right)/E^0_{\mathbf{k}}+1\right]$ and results in

$$H_0=\sum_{\mathbf{k}\neq 0}E^0_{\mathbf{k}}\,c^\dagger_{\mathbf{k}}c_{\mathbf{k}}\quad , \tag{3}$$

where $c^\dagger_{\mathbf{k}}$ creates a Bogoliubov quasi-particle of 2D quasi-momentum $\mathbf{k}$ and energy $E^0_{\mathbf{k}}=\left[(\epsilon^0_{\mathbf{k}})^2+2\bar{n}U_0\epsilon^0_{\mathbf{k}}\right]^{1/2}$.

To the system described by (3) we introduce a periodic modulation of the lattice depth V_0 of the form $V_0(t)=V_0\left(1+A\sin\Omega t\right)$, with $A\ll 1$. Thus, J_0 and U_0 become time dependent, and to lowest order in A we have $\epsilon_{\mathbf{k}}(t)=\epsilon^0_{\mathbf{k}}(1+B\sin\Omega t)$ and $U(t)=U_0(1+C\sin\Omega t)$, where B and C are calculated from the relations below Eq. (1). The resulting Hamiltonian is

$$H=\sum_{\mathbf{k}\neq 0}\left[\frac{E_{\mathbf{k}}(t)}{2}\left(c^\dagger_{\mathbf{k}}c_{\mathbf{k}}+c^\dagger_{-\mathbf{k}}c_{-\mathbf{k}}\right)+\frac{V_{\mathbf{k}}(t)}{2}\left(c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}}+\text{h.c.}\right)\right] \tag{4}$$

where $E_{\mathbf{k}}(t)=E^0_{\mathbf{k}}+\Delta E_{\mathbf{k}}(t)$, with $\Delta E_{\mathbf{k}}(t)=\left(\epsilon^0_{\mathbf{k}}/E^0_{\mathbf{k}}\right)\left[B\left(\epsilon^0_{\mathbf{k}}+\bar{n}U_0\right)+C\bar{n}U_0\right]\sin\Omega t$, and

$$V_{\mathbf{k}}(t)=\frac{\epsilon^0_{\mathbf{k}}}{E^0_{\mathbf{k}}}\bar{n}U_0(C-B)\sin\Omega t\quad . \tag{5}$$

The Hamiltonian (4) is no longer diagonal, and a time-dependent term creating pairs of excitations with opposite momenta is introduced, responsible for the parametric resonance. Following the method of Tozzo *et al.*, we introduce the operators $\alpha_{\mathbf{k}}\equiv c_{\mathbf{k}}\exp\left[i\int dt' E_{\mathbf{k}}(t')\right]$ and the Heisenberg equations on (4) yield (we set $\hbar=1$)

$$\begin{aligned}\dot{\alpha}_{\mathbf{k}}&=\gamma_{\max}\,\alpha^\dagger_{-\mathbf{k}}\left[e^{-i(\Omega t-2\int E_{\mathbf{k}}dt)}-e^{i(\Omega t+2\int E_{\mathbf{k}}dt)}\right]\\ \dot{\alpha}^\dagger_{-\mathbf{k}}&=\gamma_{\max}\,\alpha_{\mathbf{k}}\left[e^{i(\Omega t-2\int E_{\mathbf{k}}dt)}-e^{-i(\Omega t+2\int E_{\mathbf{k}}dt)}\right]\end{aligned} \tag{6}$$

where $\gamma_{\max}=(C-B)\epsilon^0_{\mathbf{k}}\bar{n}U_0/2E^0_{\mathbf{k}}$. To lowest order in A we may replace $E_{\mathbf{k}}$ by its time independent part $E^0_{\mathbf{k}}$.

When the arguments of the exponents get large, the fast oscillations effectively kill the time-dependence of $\alpha_{\mathbf{k}}$. In the regime $\Omega t\gg 1$ this is almost always the case, except on the slowly oscillating modes with $\left|\Omega-2E^0_{\mathbf{k}}\right|t\ll 1$ which fulfill the single second order equation

$$\ddot{\alpha_{\mathbf{k}}}+i\left(\Omega-2E^0_{\mathbf{k}}\right)\dot{\alpha_{\mathbf{k}}}-\gamma^2_{\max}\alpha_{\mathbf{k}}=0 \tag{7}$$

with solution $\alpha_{\mathbf{k}}=\eta^+_{\mathbf{k}}e^{i\omega_+ t}+\eta^-_{\mathbf{k}}e^{i\omega_- t}$, where

$$\omega_\pm=-\frac{1}{2}\left[(\Omega-2E^0_{\mathbf{k}})\pm\sqrt{(\Omega-2E^0_{\mathbf{k}})^2-4\gamma^2_{\max}}\right]\ . \tag{8}$$

Here $\eta^\pm_{\mathbf{k}}$ are set by the initial conditions $\alpha_{\mathbf{k}}(t=0)=c_{\mathbf{k}}(t=0)$ and $\dot{\alpha}_{\mathbf{k}}(t=0)=\gamma_{\max}\,c^\dagger_{-\mathbf{k}}(t=0)$ and result in coherent superpositions of $c_{\mathbf{k}}(t=0)$ and $c^\dagger_{-\mathbf{k}}(t=0)$. In a narrow window of energies $\left|\Omega/2-E^0_{\mathbf{k}}\right|<|\gamma_{\max}|$ the frequencies $\omega_\pm$ gain an imaginary part and the occupation of the $\eta^+_{\mathbf{k}}$ modes grows exponentially with time (resonant modes), corresponding to the resonant excitation of pairs of Bogoliubov modes with opposite momenta, whose total energy is approximately equal to the external modulation frequency. Outside this window, the time evolution of the occupation is periodic. The occupation of the Bogoliubov modes, $\langle c^\dagger_{\mathbf{k}}(t)c_{\mathbf{k}}(t)\rangle$, is then

$$\begin{aligned}&\left(\frac{\gamma_{\max}}{\gamma}\right)^2\sinh^2(\gamma t)\left[2n_{\mathbf{k}}(T)+1\right]+n_{\mathbf{k}}(T)\\ &\left(\frac{\gamma_{\max}}{\gamma}\right)^2\sin^2(\gamma t)\left[2n_{\mathbf{k}}(T)+1\right]+n_{\mathbf{k}}(T)\end{aligned} \tag{9}$$

in the regimes $4\gamma^2_{\max}>(\Omega-2E^0_{\mathbf{k}})^2$ and $4\gamma^2_{\max}<(\Omega-2E^0_{\mathbf{k}})^2$ respectively, where $\gamma\equiv\left[\left|\gamma^2_{\max}-\frac{1}{4}(\Omega-2E^0_{\mathbf{k}})^2\right|\right]^{1/2}$ and $n_{\mathbf{k}}(T)=\left[e^{E^0_{\mathbf{k}}/k_{\mathrm{B}}T}-1\right]^{-1}$ is the Bose distribution. This result is correct as long as the occupation of the excited modes (the condensate depletion) is small such that (2) and (4) hold. The exponential growth of the resonant-modes occupation is compatible with the result of Tozzo et al. [16].

In order to calculate the energy absorbed by the system due to the time dependent modulation of the lattice depth, we consider a modulation that is turned on at $t=0$ and switched off at time $t=\tau$, bringing back the Hamiltonian of the system to (3). The energy absorbed by the system during the time $0<t<\tau$ is

$$\mathcal{E}(\tau)=\sum_{\mathbf{k}\neq 0}E_{\mathbf{k}}\left[\langle c^\dagger_{\mathbf{k}}(\tau)c_{\mathbf{k}}(\tau)\rangle-\langle c^\dagger_{\mathbf{k}}(0)c_{\mathbf{k}}(0)\rangle\right] \tag{10}$$

where we perform a thermal average with the unperturbed Hamiltonian H_0. The absorption is then obtained by inserting (9) into (10), with the summation over $\mathbf{k}$ limited to the range of energies where oscillations are slow.

The absorption is strongly affected by the density of states (DOS) of the system, which is crucially dependent on the shell structure. Indeed, if the width of the ring L is much smaller than its length L_x, transverse quantization of the 2D momenta yields a spectrum composed of separate 1D branches, labelled by the integers m. The m^{th} branch has a dispersion

$$E^0_{k,m} \simeq c\left[k^2 + (\frac{\pi m}{L})^2\right]^{1/2} \quad (11)$$

where k is the 1D wavenumber along the ring and we considered the low-energy phononic dispersion with sound velocity $c = \left(2\bar{n}U_0J_0l^2\right)^{1/2}$. The DOS is characterized by Van-Hove singularities near $k = 0$ of the form $\nu(E) \sim E/\left[E^2 - E^2_{g,m}\right]^{1/2}$ where $E_{g,m} = E^0_{k=0,m}$. In addition, in the presence of the SF rings, 1D surface modes exist at the interfaces between domains, with a linear dispersion at low momenta [10]. Their DOS is however smooth and shows no singularity, which makes their detection difficult with the technique proposed here.

We now examine the dependence of the energy absorption (10) on the modulation frequency Ω. The typical parameters used in recent experiments [13] are $\tau \sim 30$ ms and $\Omega \leq 8$ kHz such that $\Omega\tau \gg 1$ and the occupation of the modes is determined by Eq. (7). At a given Ω, modes with energies in a "resonant window" of width $2\gamma_{\max}$ around $\Omega/2$ are exponentially populated in time. Thus, by varying Ω and considering the energy absorbed by the system we can get information about its spectrum. For example, in the presence of SF rings, changing Ω in the vicinity of $2E_{g,1}$ the resonant window moves across the van Hove singularity and the absorption shows a peak. The absorption is then suppressed as Ω is further increased past the singularity. A similar enhancement is predicted every time the modulation frequency meets the condition $2(E_{g,m} - |\gamma_{\max}|) < \Omega < 2(E_{g,m} + |\gamma_{\max}|)$ for some m. Fig. 1 shows the energy absorbed by the system versus Ω for a modulation of amplitude $A \sim 0.001$ in the lattice intensity V_0. For a typical $J_0 \approx U_0/45 \approx 0.4$ KHz (characteristic to the $\bar{n} = 2$ MI lobe) this corresponds to the amplitudes $B = 0.01$ in the hopping strength and $C \sim 0.001$ in the on-site repulsion.

The plot in Fig. 1 is the main result of this paper; a measurement of the energy absorption as function of the modulation frequency supplies a direct evidence for the existence of narrow SF shells, as can be seen from the difference between the dashed and the solid lines. We point out that the enhanced absorption shown in the latter is a result of the divergence in the DOS together with the fact that the exponent $\gamma_{\max}$ does not vanish at $k = 0$. In the low-energy regime $2\bar{n}U_0 \gg \epsilon^0_{k,m}$ we get $\gamma_{\max} \simeq (C - B)E^0_{k,m}/4$ which is finite for $k = 0$ if $m \neq 0$.

The energy difference between the transverse branches resulting from one SF ring, $\Delta E = E_{g,m} - E_{g,m-1} \simeq \pi c/L$ is inversely proportional to its width L. Therefore the spacing between adjacent absorption peaks in Fig. 1 is a signature of the ring-like SF domains, and their typical width may be estimated from it, given J_0, U_0 and l.

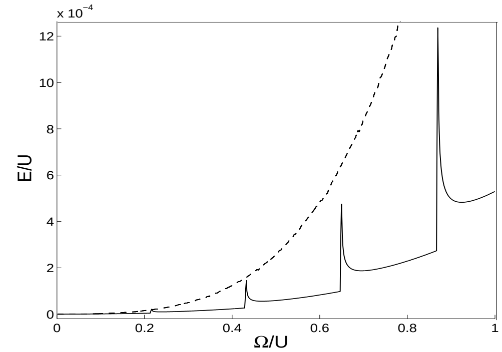

FIG. 1: The total energy absorption after a modulation of the lattice intensity V_0 with $A = 0.001$ during 10 msec, plotted as function of the modulation frequency Ω/U_0 for the mixed MI-SF regime at $U_0/J_0 = 45$ (solid line) for the inner ring, of average density $\bar{n} = 1.5$ particles per site, and for the 2D SF phase at $U_0/J_0 = 20$ (dashed line) and average density of $\bar{n} = 1.5$. We considered l=257 nm and $L = 10\,l$.

This result may be compared with the expected energy absorption in the absence of SF rings. In case J_0/U_0 is very small, the width of the SF rings gets smaller than one lattice spacing, and the SF part of the system effectively disappears. Then, the only modes that can be excited are above the MI gap and no signature appears in low energy absorption. In the opposite regime, when J_0/U_0 is larger than the critical ratio for the MI-SF transition, the entire 2D system is SF with a smooth DOS. The dashed line in Fig. 1 shows the energy absorption as function of the modulation frequency in this regime.

For a 3D cloud of atoms the arguments above hold in essentially the same form. Again, the presence of SF shells with thickness L yield several 2D Bogoliubov branches with interband separation proportional to $1/L$. In this case, the 2D nature of the modes does not lead to van Hove singularities but rather to a staircase profile in the DOS. As a result, the absorption vs. Ω will not show peaks, as in Fig. 1, but steps, from the existence and spacing of which one can infer the presence of domains and their size. Again, in contrast, a pure 3D superfluid phase would show a smooth absorption spectrum.

We now analyze the consistency of our results and their applicability to experiments. We choose the parameter values to be in a regime close to that described in [5]. The recoil energy for Sodium atoms in an optical lattice of wavelength $2l = 514$ nm is $E_R = 2\pi\hbar \times 32$kHz. For the generation of two MI rings we need $V_0 \simeq 15\,E_R$, yielding $U_0 \simeq 0.4\,E_R$ and $J_0 \simeq 0.01\,E_R$. The single band BH approximation holds, since $\frac{1}{2}U_0\bar{n}(\bar{n}-1) \ll \hbar\nu$, where $\hbar\nu = \sqrt{4E_RV_0}$ is the excitation energy to the first excited Bloch band. A typical ring of width $L \simeq 10\,l$ and $\bar{n} = 1.5$

results in a transverse gap of $E_{g,1} \simeq 0.1\, U_0$, allowing to probe about 10 transverse branches below the MI gap U_0. The visibility of the absorption peaks is strongly related to the duration of the parametric modulation. An estimate for the minimal duration required for good resolution of the peaks is $\tau \sim \frac{1}{2}\hbar\gamma_{\max}^{-1}(E_{g,1}) \sim 10\,\mathrm{msec}$, feasible in current experimental setups.

The enhancement in the population of modes at zero temperature is triggered by the presence of the zero point fluctuations, that supply a "quantum seed". At finite temperature these are augmented by thermal fluctuations. A second important fact enabling this amplification is the bosonic character of the atoms. If we consider the "fermionic version" of our hamiltonian (2), i.e. the BCS mean-field one, and proceed with the periodic modulation, no exponential growth of the occupation is expected due to the fermionic nature of the excitations.

The sharp peaks in Fig. 1 result from the confinement of the superfluid to narrow and long strips and from the long-living excitation modes that characterize superfluids. To separate the contribution of these two sources, we consider the effect of viscosity on the parametric resonances in a narrow fluid strip. We concentrate on the parametric excitation of sound waves when the sound velocity is modulated through a periodic variation of the effective mass $M(t) = M_0(1 + B\sin\Omega t)$ and the interaction constant $U(t) = U_0(1 + C\sin\Omega t)$. The equation of motion for the density fluctuation is

$$\ddot{\rho_k} + \zeta_k(t)\dot{\rho_k} + \omega_k^2(t)\rho_k = 0 \qquad (12)$$

where $\omega_k^2(t) = c_s(t)k$, $c_s(t) \propto \sqrt{U(t)/M(t)}$, $\zeta(t) \equiv \dot{M}(t)/M(t) + \eta k^2$, and η is the fluid viscosity. The transformation $\rho_k(t) = y_k(t)\exp\left[-\int_0^t \zeta_k(t')/2dt'\right]$ maps Eq. (12) onto a parametrically excited oscillator

$$\ddot{y_k} + \tilde{\omega}_{k,0}^2\left[1 + \tilde{A}\sin\Omega(t - t_0)\right]y_k = 0 \qquad (13)$$

where $\tilde{A} = \left[[C - (1 - \Omega^2/2\omega_{k,0}^2)B]^2 + (\eta k^2\Omega B/2\omega_{k,0}^2)^2\right]^{1/2}$ and $\tilde{\omega}_{k,0}^2 = \omega_{k,0}^2 - (\eta k^2/2)^2$ to lowest order in B and C. The resonant $y_k's$ are exponentially enhanced, $y_k(t) \sim \exp[\tilde{\gamma}_{max}t]$ with $\tilde{\gamma}_{max} \simeq \frac{1}{4}\tilde{A}\tilde{\omega}_{k,0}$. The resulting dynamics of the sound modes has the form $\rho_k \sim \exp[(\tilde{\gamma}_{max} - \eta k^2/2)t]\exp[-B\sin\Omega t/2]$.

Density fluctuations will grow exponentially in time only if the amplitude of the modulation is large enough such that $\tilde{\gamma}_{max}$ is real and satisfies $\tilde{\gamma}_{max} > \eta k^2/2$. Furthermore, the width of the energy absorption peaks is $2\tilde{\gamma}_{max}$. Thus, when the amplitude of the modulation is large enough for the formation of the peaks, the peaks acquire a finite width, which for the m'th peak must be larger than $\eta(\pi m)^2/2L^2$. When the viscosity is large enough such that the width of the peaks is larger than their spacings, the peaks are fully smeared.

In conclusion, we considered an ultracold cloud of Bose atoms subjected to a periodically modulated optical lattice which induces parametric amplification of its excitations. The energy absorption as a function of the modulation frequency may serve as a tool to detect features in the DOS of the system, that are induced by a transition from a uniform SF phase for weak lattice intensities to a mixed MI-SF regime for stronger ones. This enables the experimental detection of the coexistence of MI-SF domains and the measurement of their typical size. In addition, we demonstrated the broadening of absorption peaks due to damping in normal fluids which is absent in case the domains are superfluids.

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